

## USING COLUMN GENERATION TO SOLVE AN INDUSTRIAL MIXING PROBLEM.

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**Abstract.** In this paper, we formulate and solve a real life coal blending problem using a Column Generation Approach. The objective of the model is to prescribe optimal mixes of coal used to produce coke. The problem is formulated as a mixed integer program. It involves various types of constraints arising out of technical considerations of the blending process. The model also incorporates integer variables. Three heuristics based on column generation ideas are proposed to solve this problem. The heuristics enabled by the use of dual variables related to the ratio amounts of each coal.

**Mathematics Subject Classification.** 90B30, Operations research, mathematical programming; Production models , 49M27 Decomposition methods

### 1. INTRODUCTION

In this paper, we consider a real life problem: a coal blending problem for coke production. The model includes not only quality constraints for the coke but also all the constraints coming from the different transportation modes and the inventory constraints.

Different types of constraints should be applied during the production. Firstly, the coke is subject to a collection of *quality constraints* which, for some of them, may depend on the clients. These constraints are all linear. But there are also,

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for technical reasons, a *maximum number of coal types in the mix*, this number depending on the number of gates of the plant. A *mix* is simply a result of running for a certain duration using particular ratios of different coal types.

There is also, for technical reasons, a *minimal proportion for each coal present* in the mix. As we shall see in Section 3, the modeling of these conditions introduces *integer variables* and *nonlinear constraints* which may be linearized under a restrictive condition (See Williams [9]).

Thirdly, the coke delivered to the customers can be obtained using different mixes of coals. However, the total number of mixes used by a plant in a given period is bounded from above: there is an important fixed cost to change the mix and generally, two different mixes can be used each month. Finally, *capacity constraints* are present at the plants, as well as *minimum production levels*.

The industrial problem tackled in this paper is more difficult than the classical coal blending problem of the Operations Research literature considered by Sarker and Gunn [6] or Greenberg [4] because of the *presence of binary variables*. These binary variables are required to model the minimum and/or maximum level of use of each coal. Sarker and Gunn [6] solve the tactical planning/coal blending problem. This leads to a nonlinear nonconvex problem for which solutions are computed using simple Successive Linear Programming. They used the Lasdon SLP implementation based on the XLP code of Marsten (See Lasdon [5]). Greenberg [4] looks at the same pooling problem that arises in blending materials but considers a different application, namely the blending of crude or refined petroleum. He presents a new method based upon computational geometry which provides exact answers to questions of sensitivity analysis for this nonlinear nonconvex

We solve a more complicated problem through the following column generation technique (See Vanderbeck and L. Wolsey [7] for an exact algorithm for IP column generation, Vanderbeck [8] for the branch-and-price algorithm or Desaulniers, Desrosiers, and Solomon [2] for a didactic introduction to the use of column generation technique in integer programming). For each time period, and each plant, by fixing the coal delivery prices at the entry of the plant, i.e. a price that represents the unit purchasing and transportation costs, we determine the *optimal coal mix for one unit of production*. This constitutes our *subproblems*. Then, the *Master problem* determines the level of use of these mixes. The Master problem also determines the coal delivery plan to the plants and the coke delivery plan to the clients. As we shall see in Section 4, the solution of the Master problem leads to reviewing the delivery prices at the entry of the plant for each coal and a new mix based on these new prices is generated by solving the subproblems. Consequently, new columns are generated for the Master problem. Note that there are also integer variables at the upper level, since only a limited number of different mixes can be considered at each time period.

The *main contribution of this paper* is to solve a real life problem coming out of the industry which is a large-scaled problem intractable by classical operation research methods. Defining three heuristics methods based on column generation techniques, we furnished reasonable solutions to the industrial company.

## 2. PROBLEM DESCRIPTION

The manager of a coke production company wants to determine the operation schedule for the next three months for the five production units situated in Belgium. We use  $t$  as index for the time periods. The three time periods correspond to the months January to March. We note  $days_t$  the number of days for month  $t$ .

These five plants produce different types of coke by mixing different types of coals and then firing them to produce coke. Different type of clients corresponding to different type of steel are furnished by the company which results in different mixes of coal types.

The demand is assumed to be known with accuracy for the first three months of the year. The different coals are purchased from several countries and brought to Belgium by means of two transportation modes, namely by boat or by rail. The buying prices of the different coals and the transportation cost from producer to the plant are known.

### 2.1. PLANTS DESCRIPTION

Each production plant  $k$  is characterized by a *daily capacity of plant*  $cap_k$  (See Table 1), namely the maximum amount of coal that can be handled by the plant.

| $k$ | $cap_k$ | $minuse_k$ | $gates_k$ | $mincoal_k$ | $maxcoal_k$ |
|-----|---------|------------|-----------|-------------|-------------|
| 1   | 2 300   | 60 %       | 8         | 5 %         | 100 %       |
| 2   | 2 850   | 60 %       | 4         | 15 %        | 35 %        |
| 3   | 1 250   | 75 %       | 8         | 10 %        | 100 %       |
| 4   | 3 500   | 60 %       | 8         | 10 %        | 100 %       |
| 5   | 3 500   | 60 %       | 8         | 10 %        | 100 %       |

TABLE 1. Plant characteristics

Since the plant can never be stopped, there is also a *minimal utilization rate for each plant*, denoted  $minuse_k$  for plant  $k$ . To put the coals into the oven, there is only a limited number of entry gates, denoted  $gates_k$  for plant  $k$ . This induces that the number of coal types in the mix is limited. There are also a *minimal* and a *maximal proportion* for each coal in the mix, denoted respectively  $mincoal_k$  and  $maxcoal_k$  for plant  $k$ . Note that these two parameters are function of the plant and are independent of the coal type: in fact, the minimal and maximal proportion depend on the charging machinery of the plants. Since the plants were constructed at different time periods, the technology used is different, which explains the difference from plant to plant.

Plants also differ by their unitary production cost. We note by  $prodcost_{kt}$  the unitary production cost of plant  $k$  at time period  $t$  (See Table 2).

| $k$ | $prodcost_{k1}$ | $prodcost_{k2}$ | $prodcost_{k3}$ |
|-----|-----------------|-----------------|-----------------|
| 1   | 11.450          | 11.400          | 11.350          |
| 2   | 18.175          | 18.100          | 18.025          |
| 3   | 38.150          | 38.000          | 37.850          |
| 4   | 24.125          | 24.025          | 23.925          |
| 5   | 11.300          | 11.250          | 11.200          |

TABLE 2. Production cost (euro per ton of coal)

## 2.2. COAL CHARACTERISTICS

The firm can use during this production horizon 16 types of coal which differ by their characteristics such as the *ash rate*, denoted  $ash_c$  for coal  $c$ , the *sulfur rate*, denoted  $sulf_c$ , the *alkali rate*, denoted  $alk_c$ , the *volatile part*, denoted  $vol_c$  and the *wet part*, denoted  $wet_c$ . The wet part is the unusable part of the coals. One also distinguishes three classes among the coals: the “High Volume”, denoted HV, the “Mid Volume”, denoted MV, and the “Low Volume”, denoted LV (See Table 3).

| $c$ | $ash_c$ | $sulf_c$ | $alk_c$ | $vol_c$ | $wet_c$ | Type |
|-----|---------|----------|---------|---------|---------|------|
| 1   | 4.99 %  | .85 %    | .12 %   | 17.89%  | 6.63%   | LV   |
| 2   | 5.53%   | .70%     | .09%    | 17.48%  | 7.77%   | LV   |
| 3   | 7.72%   | .94%     | .31%    | 22.71%  | 10.15%  | LV   |
| 4   | 8.30%   | .67%     | .14%    | 21.00%  | 8.00%   | LV   |
| 5   | 8.07%   | .70%     | .16%    | 23.60%  | 9.47%   | MV   |
| 6   | 4.83%   | .88%     | .16%    | 30.19%  | 6.80%   | HV   |
| 7   | 6.86%   | 1.15%    | .24%    | 29.90%  | 7.54%   | HV   |
| 8   | 6.14%   | .84%     | .22%    | 31.39%  | 8.66%   | HV   |
| 9   | 6.01%   | .82%     | .19%    | 32.75%  | 6.29%   | HV   |
| 10  | 6.45%   | .88%     | .17%    | 33.08%  | 7.20%   | HV   |
| 11  | 7.44%   | .66%     | .24%    | 27.43%  | 6.45%   | HV   |
| 12  | 7.70%   | .71%     | .23%    | 32.09%  | 6.81%   | HV   |
| 13  | 7.00%   | .98%     | .32%    | 25.10%  | 8.00%   | MV   |
| 14  | 7.60%   | .57%     | .18%    | 19.30%  | 10.14%  | LV   |
| 15  | 5.79%   | .82%     | .18%    | 24.39%  | 8.99%   | MV   |
| 16  | 5.30%   | .72%     | .13%    | 33.50%  | 7.90%   | HV   |

TABLE 3. Coal characteristics

## 2.3. AVAILABILITY, DELIVERY AND DEPARTURE PRICES OF THE COALS

Some of the coals come *by boat* (See Figure 1). We use the subset  $b$  of  $c$  to note these coals. They mainly come from North America and from Australia. There are *two possible arriving harbors* that we index by  $h$ . There is an *initial coal inventory at the beginning of the year in the two harbors*, noted by  $initstock_{ch}$  (See Table 4). We note  $expq_{ct}$  deliveries correspond to already ordered quantities *expected for some of these coals*. The landing harbor is not yet decided.

| Coal<br>( $c$ ) |     | $initstock_{ch}$ (tons) |         | $expq_{ct}$ (tons) |         |         |    | price  |
|-----------------|-----|-------------------------|---------|--------------------|---------|---------|----|--------|
|                 |     | $h = 1$                 | $h = 2$ | $t = 1$            | $t = 2$ | $t = 3$ |    |        |
| 1               | $b$ | 558                     | 7 620   | 70 000             | 70 000  | 100 000 | \$ | 46.80  |
| 2               | $b$ |                         |         |                    |         |         | \$ | 46.75  |
| 3               | $t$ |                         |         | 26 000             | 26 000  | 26 000  | €  | 55.65  |
| 4               | $b$ |                         | 36 655  |                    |         |         | \$ | 37.75  |
| 5               | $b$ | 42 760                  | 22 570  | 40 000             |         | 60 000  | \$ | 45.75  |
| 6               | $t$ |                         |         | 49 600             | 49 600  | 49 600  | €  | 42.225 |
| 7               | $t$ |                         |         | 14 800             | 14 800  | 14 800  | €  | 42.225 |
| 8               | $t$ |                         |         | 10 000             | 10 000  | 10 000  | €  | 41.775 |
| 9               | $t$ |                         |         | 20 000             | 20 000  | 20 000  | €  | 43.575 |
| 10              | $b$ |                         |         |                    |         |         | \$ | 46.65  |
| 11              | $b$ | 5 950                   |         |                    |         |         | \$ | 49.25  |
| 12              | $b$ |                         |         |                    |         |         | \$ | 44.10  |
| 13              | $t$ |                         |         | 16 000             | 16 000  | 16 000  | €  | 55.125 |
| 14              | $b$ | 43 505                  |         | 40 000             |         | 60 000  | \$ | 40.00  |
| 15              | $t$ |                         |         | 42 400             | 42 400  | 42 400  | €  | 44.95  |
| 16              | $b$ |                         | 6 450   | 25 000             | 25 000  |         | \$ | 46.80  |
|                 |     | Currency rate           |         | $t = 1$            | $t = 2$ | $t = 3$ |    |        |
|                 |     | \$                      |         | 0.975              | 0.925   | 0.875   |    |        |
|                 |     | €                       |         | 1                  | 1       | 1       |    |        |

TABLE 4. Initial inventory, expected quantities and coal prices at the departure

The other coals are delivered *by rail*. They mainly come from Germany and from Eastern Europe. There is no storage capacity for these coals. Deliveries, noted  $expq_{ct}$ , are also expected for these coals. The final destination of these quantities is not yet decided. (See Table 4).

Some of the coals are paid in US dollar (\$), others in Euro (€). Expected exchange rate for the next three months are given by Table 4. The \$ is the

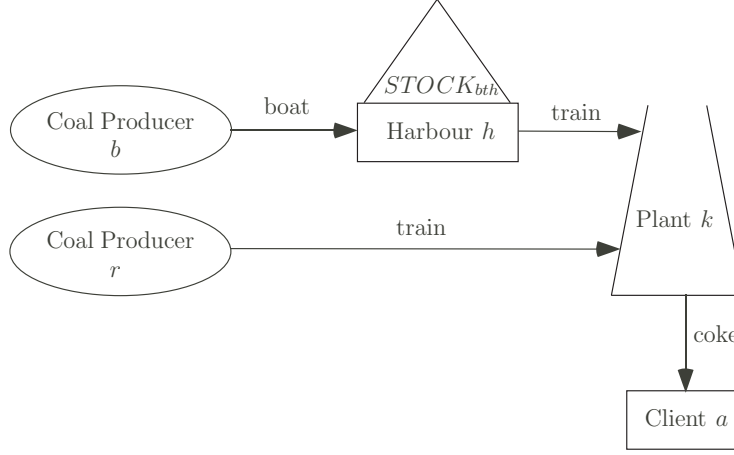


FIGURE 1. Supply chain for the input to the plants

reference currency for the coals coming by boat. The € is used for the other coals. Prices are given at the harbour or station of departure in Table 4 where the relevant currency is also given for each coal. Multiplying the price in foreign currency by the expected exchange rate, we obtain the price of coal  $c$  at time period  $t$  in € denoted  $price_{ct}$ .

#### 2.4. THE TRANSPORTATION COSTS

The *transportation costs by boat to the harbour*, denoted  $boatcost_{ch}$ , are given in \$ per ton of coal (See Table 5). There is no difference for the transportation costs between the two arrival harbors. *Handling costs at the harbour*, denoted  $dockcost_h$  are also the same at the two harbours due to the competition between the harbours: 3.3875 € per ton. *Transportation costs from harbours to plants*, denoted  $transpcost_{hk}$  are also given in € per ton (See Table 5).

Note that there is *no transportation cost between harbour 2 and plant 3*, the plant being located in the harbour area. Consequently, plant 3 is only supplied from harbour 2. The four other plants are also supplied by rail. The *transportation costs by rail between the departure station and the plants*, denoted  $railcost_{ck}$ , are given in € per ton by Table 5 for all coals that are coming directly from the production site to the plant by rail.

#### 2.5. CHARACTERISTICS OF THE DEMAND

The coke company must satisfy the demand of 13 clients for the next three months. We used  $a$  as index for the client. The *demand* of client  $a$  in time period  $t$  is denoted  $dem_{at}$  (See Table 6). In order to minimize the coke transportation

| (\$/ton) | $boatcost_{ch}$ |         |         |         |          |          |          |          |          |
|----------|-----------------|---------|---------|---------|----------|----------|----------|----------|----------|
|          | $c = 1$         | $c = 2$ | $c = 4$ | $c = 5$ | $c = 10$ | $c = 11$ | $c = 12$ | $c = 14$ | $c = 16$ |
| $h = 1$  | 5.1             | 4.5     | 10.25   | 8.15    | 3.75     | 3.30     | 0        | 7.5      | 5.10     |
| $h = 2$  | 5.1             | 4.5     | 10.25   | 8.15    | 3.75     | 3.30     | 0        | 7.5      | 5.10     |

| (\\$/ton) | $transpcost_{hk}$ |         | $railcost_{ck}$ (\\$/ton) |         |         |         |         |          |          |
|-----------|-------------------|---------|---------------------------|---------|---------|---------|---------|----------|----------|
|           | $h = 1$           | $h = 2$ | $c = 3$                   | $c = 6$ | $c = 7$ | $c = 8$ | $c = 9$ | $c = 13$ | $c = 15$ |
| $k = 1$   | 4.4675            | 4.4675  | 9.155                     | 4.67    | 4.67    | 4.67    | 9.155   | 9.155    | 4.67     |
| $k = 2$   | 2.6375            | 3.25    | 7.39                      | 4.2625  | 4.2625  | 4.2625  | 7.39    | 7.39     | 4.2625   |
| $k = 3$   | $+\infty$         | 0       | -                         | -       | -       | -       | -       | -        | -        |
| $k = 4$   | 3.86              | 3.86    | 8.29                      | 6.08    | 6.08    | 6.08    | 8.29    | 8.29     | 6.08     |
| $k = 5$   | 2.460             | 4.105   | 7.7475                    | 2.31    | 2.31    | 2.31    | 7.7475  | 7.7475   | 2.31     |

TABLE 5. Boat transportation costs, Dock cost and rail transportation cost to the plants

cost, the plant requested by each client is given. A few of them (clients 4 and 6) can be delivered by two plants, the distances being similar.

| $dem_{at}$ | $t = 1$ | $t = 2$ | $t = 3$ | Plant  |
|------------|---------|---------|---------|--------|
| Client 1   | 11500   | 12600   | 12600   | 3      |
| Client 2   |         | 4000    | 4000    | 1      |
| Client 3   | 45000   | 45000   | 45000   | 1      |
| Client 4   | 7000    | 8000    | 8000    | 2 or 3 |
| Client 5   | 2000    | 2000    | 2000    | 3      |
| Client 6   | 16500   | 24800   | 14400   | 1 or 2 |
| Client 7   | 26700   | 19400   | 14800   | 2      |
| Client 8   |         | 3500    |         | 2      |
| Client 9   | 6000    | 6000    | 5000    | 1      |
| Client 10  | 16000   | 16000   | 14000   | 2      |
| Client 11  | 12000   | 12000   | 12000   | 3      |
| Client 12  | 58452   | 49002   | 51644   | 4      |
| Client 13  | 68516   | 68132   | 71162   | 5      |

TABLE 6. Demand for the next three months

We explain now the *quality constraints* that the coal mix must meet. The first one concerns the *volatile rate*. The *volatile rate of the mix must be between 24 %*

| Client   | $maxash_a$ | $minsulf_a$ | $maxsulf_a$ | $maxalk_a$ | $minlv_a$ | $maxlv_a$ |
|--|------------|-------------|-------------|------------|-----------|-----------|
| Client 1   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 2   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 3   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 4   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 5   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 6   | 9.5        | .7          | .9          | .30        | 30.       | 100.      |
| Client 7   | 9.5        | .7          | .9          | .30        | 30.       | 100.      |
| Client 8   | 10.        |             | 1.          | .30        | 40.       | 50.       |
| Client 9   | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 10  | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 11  | 10.        |             | 1.          | .30        | 30.       | 100.      |
| Client 12  | 9.5        | .7          | .9          | .30        | 30.       | 100.      |
| Client 13  | 9.5        | .7          | .9          | .30        | 30.       | 100.      |
| Multiplicative coefficient<br>from the mix to the coke |            |             | Ash         | Sulfur     |           | Alkali    |
|  |            |             | 1.32        | 0.92       |           | 1.32      |

TABLE 7. Specifications which depend on the client.

and 26 %. We denote these two parameters as  $minvol$  and  $maxvol$ . The second one concerns the *ash rate*. The mix has a *maximal ash rate* depending on the client. We note  $maxash_a$  this quantity for client  $a$ . The third specification concerns the *sulfur rate*. There is a *lower* and *upper limit on sulfur*, denoted  $minsulf_a$  and  $maxsulf_a$ , which also depends on the client. The fourth specification concerns the *alkali rate*. There is an *upper limit on alkali*, denoted  $maxalk_a$ , which also depends on the client  $a$ . The fifth specification concerns the *Low Volume rate*. The *lower* and *upper limits on Low Volume*, denoted  $minlv_a$  and  $maxlv_a$ , also depend on the client (See Table 7). For three characteristics (ash, sulfur, alkali), there is a multiplicative coefficient from the mix to the coke: they are denoted respectively  $multash$ ,  $multsulf$  and  $multalk$  (See Table 7).

For all minimal and/or maximal characteristic rates that depend on the client, we shall consider as specification for the mix of the plant *the specification of the most restrictive client* that the plant has to serve.

The sixth specification concerns the *rate in "Mid Volume"*. Here there are *lower* and *upper bounds on Mid Volume*, denoted  $minmv$  (0.25) and  $maxmv$  (0.25) respectively, which do not depend on the client. In fact, for technical reasons, there is a fixed rate 1/4 of Mid volume coals. The seventh specification concerns the *soft rate*. There is an *upper bound on Soft coal*, denoted  $maxsoft$  (0.10), which does not depend on the client. The only soft coal is coal 12. The eighth specification concerns the *Australian coal rate*. Here, there is for commercial reason an *upper*



*bound*, denoted *maxaus* (0.30), which does not depend on the client. The only Australian coal is coal 4.

Finally, one must account for the fact that the number of mixes that can be used for each plant per period is limited. This maximum number of mixes, denoted *maxmix<sub>kt</sub>*, is equal to 2 mixes per month for each plant. The different mixes will be indexed by *m*.

### 3. PROBLEM FORMULATION

We have already introduced the following indices : the coals are indexed by *c*, the plants by *k*, the time periods by *t*, the mixes by *m*, the harbours by *h* and the clients by *a*. For facility of the notation, we have also introduced the subindices *b* for the coals coming by boat and *r* for the coals coming by rail.

#### 3.1. CHOICE OF DECISION VARIABLES

The first set of variables concerns the **production of coke in the different plants**: we denote by *COAL<sub>cktm</sub>* the quantity of coal *c* used in plant *k* at the time period *t* in the mix *m*. We introduce only for convenience the variable *SUM<sub>ktm</sub>* which is the total quantity of coals for plant *k* in the mix *m* at the time period *t*. These two quantities are given in tons. To account for the limited number of coal in the mix, we introduce the binary variables *COALPRES<sub>cktm</sub>* to indicate the fact that coal *c* is present in the mix *m* for plant *k* at time period *t*. To account for the limited number of mixes at each time period, we introduce the binary variables *MIXPRES<sub>ktm</sub>* to indicate the fact that mix *m* is used for plant *k* at time period *t*. A second set of variables is used to **manage the deliveries and inventories**

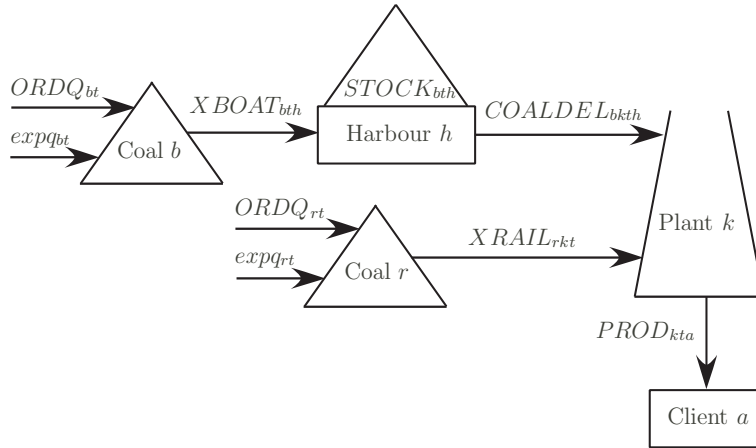


FIGURE 2. Choice for the decision variables for the coal deliveries and inventories

**of coals** (See Figure 2).

We denote by  $ORDQ_{ct}$  the new ordered quantity of coal  $c$  at time period  $t$ . We assume that the orders placed in period  $t$  are also delivered in period  $t$ . Thus, this quantity addition to the expected quantities at time period  $t$ ,  $expq_{ct}$  which are given (See Section 2). We denote by  $XBOAT_{bth}$  the quantity of coal  $b$  coming by boat that is sent to harbour  $h$  at time period  $t$  and by  $XRAIL_{rkt}$  the quantity of coal  $r$  coming by rail that is sent to coke plant  $k$  at time period  $t$ . We denote by  $STOCK_{bth}$  the inventory of coal delivered by boat  $b$  at the end of time period  $t$  in harbour  $h$ . We denote by  $COALDEL_{bkt}$  the coal  $b$  deliveries from harbour  $h$  to plant  $k$  at time period  $t$ . We denote by  $PROD_{kta}$  the tons of coke produced in plant  $k$  at time period  $t$  for client  $a$ . These variables are necessary to compute the coke transportation plan since some clients can be supplied by more than one plant.

### 3.2. MATHEMATICAL EXPRESSION OF THE OBJECTIVE

The objective function is the sum of six following terms:

$$\min z = \sum_c \sum_t price_{ct}(expq_{ct} + ORDQ_{ct}) \quad (1)$$

$$+ \sum_b \sum_t \sum_h (boatcost_{bh} + dockcost_h) XBOAT_{bth} \quad (2)$$

$$+ \sum_b \sum_t \sum_h rate \cdot delprice_{bth} STOCK_{bth} \quad (3)$$

$$+ \sum_b \sum_k \sum_t \sum_h transpcost_{hk} COALDEL_{bkt} \quad (4)$$

$$+ \sum_r \sum_k \sum_t railcost_{rk} XRAIL_{rkt} \quad (5)$$

$$+ \sum_k \sum_t \sum_m prodcost_{kt} SUM_{ktm} \quad (6)$$

- the coal *purchasing cost* (1) where  $price_{ct}$  is the coal price  $c$  at period  $t$ ,
- the *boat transportation cost* and the *handling cost* at the harbors (2) where  $boatcost_{bh}$  is the boat transportation cost and  $dockcost_h$  is the handling cost at harbor,
- the *holding inventory cost* for each coal stored at the arrival harbor (3) where  $rate$  is the monthly opportunity rate (0,5 %) and  $delprice_{bth}$  is the delivery price, i.e the sum of the purchasing cost, the boat transportation cost and the handling cost at the harbor:

$$delprice_{bth} = price_{bt} + boatcost_{bh} + dockcost_h$$

- the *transportation cost from harbor to plants* (4) where  $transpcost_{hk}$  is the unitary transportation cost from the harbour to the plant

- the *transportation cost by rail* (5) with  $railcost_{rk}$  the unitary rail cost,
- the *production costs* (6) with  $prodcost_{kt}$  the unitary production cost.

### 3.3. MATHEMATICAL EXPRESSION OF THE CONSTRAINTS

We have the following relations between theses variables:

$$SUM_{ktm} = \sum_c COAL_{cktm} \quad (7)$$

$$COAL_{cktm} \leq COALPRES_{cktm} cap_k days_t \quad (8)$$

$$SUM_{ktm} \leq MIXPRES_{ktm} cap_k days_t \quad (9)$$

$$minuse_k cap_k days_t \leq \sum_m SUM_{ktm} \leq cap_k days_t \quad (10)$$

$$expq_{bt} + ORDQ_{bt} = \sum_h XBOAT_{bth} \quad (11)$$

$$expq_{rt} + ORDQ_{rt} = \sum_k XRAIL_{rkt} \quad (12)$$

$$\sum_h COALDEL_{bkt h} = \sum_m COAL_{bktm} \quad (13)$$

$$XRAIL_{rkt} = \sum_m COAL_{rktm} \quad (14)$$

$$\sum_m MIXPRES_{ktm} \leq maxmix_{kt} \quad (15)$$

$$STOCK_{bth} - STOCK_{b,t-1,h} = XBOAT_{bth} - \sum_k COALDEL_{bkt h} \quad (16)$$

$$\sum_m \sum_c (1 - wet_c) * COAL_{cktm} = \sum_a PROD_{kta} \quad (17)$$

$$\sum_{k \text{ delivering } a} PROD_{kta} \geq dem_{at} \quad (18)$$

$$\sum_c COALPRES_{cktm} \leq gates_k \quad (19)$$

$$mincoal_k COALPRES_{cktm} \leq \frac{COAL_{cktm}}{SUM_{ktm}} \quad (20)$$

$$\frac{COAL_{cktm}}{SUM_{ktm}} \leq maxcoal_k COALPRES_{cktm} \quad (21)$$

- Equation (7) computes the *sum of coals* for each mix and each time period.
- Equation (8) and (9) are the definition of the *binary variables*: namely they indicate the presence of the coal or the use of the mix. Note that  $cap_k$  is the daily coal entry capacity for plant  $k$  and  $days_t$  is the number of days in time period  $t$ . This number of days is needed since capacity is a daily capacity and the variables are monthly quantities of coal coming in.

- Equations (10) express the *minimal utilization rate* and *capacities of plants*.
- Equation (11) expresses that the total amount of coal delivered by boat must be allocated between the two harbors since equation (12) expresses that the total amount of coal delivered by rail must be allocated between the plants.
- Equation (13) is the balance for each coal coming by boat at each plant since equation (14) is the balance for each coal delivered by rail.
- Equation (15) expresses the maximal number of mixes for each plant and each time period noted  $maxmix_{kt}$ .
- Equation (16) expresses the balance at the end of time period  $t$  for coal  $b$  at each harbor  $h$  since equation (17) expresses the balance at the exit of the plants.
- Equation (18) is the demand satisfaction.
- Equation (19) expresses the maximal number of coal types in the mix.
- Equation (20) concerns the minimal use of each coal in a mix, noted  $mincoal_k$  since equation (21) concerns the maximal use of each coal in a mix, noted  $maxcoal_k$ .

We have two types of variables:

- the *continuous nonnegative variables* which must be *nonnegative*:

$$COAL_{cktm}, SUM_{ktm}, COALDEL_{bkth}, PROD_{kta}, \\ XBOAT_{bth}, XRAIL_{rkt}, ORDQ_{bt}, STOCK_{bth} \geq 0$$

- and the *binary variables*:

$$COALPRES_{cktm}, MIXPRES_{ktm} \in \{0, 1\}$$

Note also that the two last constraints (20) and (21) are nonlinear. All the other quality constraints on the mix are easy to write. They are listed in appendix A.

#### 4. SOLUTION METHOD

Except for constraints (20) and (21), all the constraints of the problem are purely linear. Note that the left hand side of constraint (20) must be multiplied by the variable  $COALPRES$ , otherwise all the coals would be present in the mix.

The solution technique followed by Sarker and Gunn [6] is to define for each ratio  $COAL_{cktm}/SUM_{ktm}$  a variable, say  $f_{cktm}$ , introducing for each  $c, k, t$  and  $m$  the constraint:

$$f_{cktm}SUM_{ktm} = COAL_{cktm}.$$

They obtain a problem with a nonlinear part of bilinear type (product of variables). Heuristics for solving such problems have been proposed in the OR literature. They are related to techniques for solving fractional linear programs and resort to successive linear programming (SLP).

But in our problem, integer variables remain since constraints (20) and (21) are written as:

$$\min_{coal_k} COALPRES_{cktm} \leq f_{cktm} \leq \max_{coal_k} COALPRES_{cktm}.$$

We propose here a heuristic method based on column generation ideas. The problem formulated in Section 3 has two types of binary variables: one called  $COALPRES_{cktm}$  to indicate the presence of coal  $c$  in the mix  $m$  for plant  $k$  at time period  $t$ . The other one, called  $MIXPRES_{ktm}$  indicates that the mix  $m$  is used for plant  $k$  at time period  $t$ . We know that column generation procedures solve only LP problems. In our heuristic method, we keep the binary variable in the subproblem (for limiting the number of coal types in the mix) and in the main program (for limiting the number of mixes used for each plant and each time period).

The *subproblems* are defined as follows. For each plant, and for each time period, initially we consider the delivery price at the entry of the plant as the sum of the coal price and the total transportation cost to the plant. We seek a mix of **one ton** that satisfies all the quality constraints on the mix. This gives an *initial column of coal proportions* for each plant for each time period. Note that imposing that the *sum of coals be equal to zero* is a simple manner to *linearize the two non linear constraints* (20) and (21).

In the *master problem*, we shall determine the level of utilization of these mixes in order to meet the demand of the several clients at minimal production, transportation and coal purchasing costs. Multiplying the level of use of a mix by the column of the coal proportions in the mix for a particular plant, we obtain the coal quantities needed for the plant at the time period.

To *update the delivery price at the entry* of the plant, we consider the *dual price of the balance at the entry of the plant constraints* (13) or (14) depending on the transportation mode. In fact, these dual variables have a marginal effect on the objective function: the increasing by one unit of the delivery of this coal at this period for the plant.

As previously said, we suppose initially for the subproblems that all the the coals were bought from the producer to the plant to compute the initial coal delivery price at the entry of the plant. One main iteration is the solution of the 15 subproblems (5 plants  $\times$  3 time periods) and of one main problem. At the end of the first main iteration, we update the initial delivery price at the entry of the plant by substituting the dual price of constraints (13) or (14).

#### 4.1. THE SUB PROGRAMS

The subproblems correspond to the following task: find, for each time period and each plant, the *optimal mix for one unit of coal mix*. The constraints encompass all the mix quality constraints, the maximal number of coal constraint and the minimal and maximal rate for each coal in the mix constraints. The only

variables are variables  $COAL_c$  and  $COALPRES_c$ . Recall that the objective coefficients encompass at the first iteration the coal purchasing cost, transportation cost, handling cost and the production cost. At the following iterations, we only consider the sum of the dual variable and the production cost. We have thus the following *sub problems*  $SP(k,t)$  where  $p(c,k,t)$  is the *delivery price of the coal c* to plant  $k$  at time period  $t$ :

$$\begin{aligned}
 \min z &= \sum_c p(c,k,t) COAL_{ckt} \\
 \text{s.t.} &\left\{ \begin{array}{ll}
 \sum_c COAL_{ckt} &= 1 \\
 \sum_c COALPRES_{ckt} &\leq gates_k \\
 mincoal_k COALPRES_{ckt} &\leq COAL_{ckt} \\
 COAL_{ckt} &\leq maxcoal_k COALPRES_{ckt} \\
 \\
 \sum_c ash_c multash COAL_{ckt} &\leq \min_{a \text{ delivered by } k} maxash_a SUM_{kt} \\
 \sum_c sulf_c multisulf COAL_{ckt} &\geq \max_{a \text{ delivered by } k} minsulf_a SUM_{kt} \\
 \sum_c sulf_c multisulf COAL_{ckt} &\leq \min_{a \text{ delivered by } k} maxsulf_a SUM_{kt} \\
 \sum_c alk_c multalk COAL_{ckt} &\leq \min_{a \text{ delivered by } k} maxalk_a SUM_{kt} \\
 \sum_c vol_c COAL_{ckt} &\geq minvol SUM_{kt} \\
 \sum_c vol_c COAL_{ckt} &\leq maxvol SUM_{kt} \\
 \sum_{c \in LV} COAL_{ckt} &\geq \max_{a \text{ delivered by } k} minlv_a SUM_{kt} \\
 \sum_{c \in LV} COAL_{ckt} &\leq \min_{a \text{ delivered by } k} maxlv_a SUM_{kt} \\
 \sum_{c \in MV} COAL_{ckt} &\leq maxmv SUM_{kt} \\
 \sum_{c \in MV} COAL_{ckt} &\geq minmv SUM_{kt} \\
 \sum_{c \text{ is a Soft Coal}} COAL_{ckt} &\leq maxsoft SUM_{kt} \\
 \sum_{c \text{ is an Australian coal}} COAL_{ckt} &\leq maxaus SUM_{kt}
 \end{array} \right.
 \end{aligned} \tag{22}$$

We need thus to solve as many problems as there are plants and time periods. This can be done by using two inner loops in GAMS/OSL (See Brooke, Kendrick and Meeraus [1]): the first one on time periods and the second one on plants. These problems are easy to solve: they have only 16 linear variables ( $COAL$ ) and

16 binary variables (*COALPRES*). We note  $coal(c, k, t, m)$  the *proportion of coal*  $c$  used in mix  $m$  (initially,  $m$  is set to 1) at plant  $k$  at time period  $t$ .

#### 4.2. THE MAIN PROBLEM

The *main problem* corresponds to the determination of the delivery planning of coals to the plants, the allocation of ordered quantities to the harbors, the determination of the level of use of the mixes generated by the sub-problems for each plant and each time period and finally the coke transportation planning to the final clients. We obtain the following *main problem*:

$$\begin{aligned}
 \min z = & \sum_c \sum_t price_{ct}(expq_{ct} + ORDQ_{ct}) + \sum_b \sum_t \sum_h rate \cdot delprice_{bth} \\
 & STOCK_{bth} + \sum_b \sum_t \sum_h (boatcost_{bh} + dockcost_h) XBOAT_{bth} \\
 & + \sum_b \sum_k \sum_t \sum_h transpcost_{hk} COALDEL_{bkth} \\
 & + \sum_r \sum_k \sum_t railcost_{rk} XRAIL_{rkt} + \sum_k \sum_t \sum_m prodcost_{kt} SUM_{ktm} \\
 \text{s.t.} \quad & \left\{ \begin{array}{ll}
 SUM_{ktm} & \leq MIXPRES_{ktm} cap_k days_t \\
 minuse_k cap_k days_t & \leq \sum_m SUM_{ktm} \leq cap_k days_t \\
 expq_{bt} + ORDQ_{bt} & = \sum_m XBOAT_{bth} \\
 expq_{rt} + ORDQ_{rt} & = \sum_h XRAIL_{rkt} \\
 \sum_h COALDEL_{bkth} & = \sum_k coal(b, k, t, m) SUM_{ktm} \\
 XRAIL_{rkt} & = \sum_m coal(r, k, t, m) SUM_{ktm} \\
 \sum_m MIXPRES_{ktm} & \leq maxmix_{kt} \\
 STOCK_{bth} - STOCK_{b,t-1,h} & = XBOAT_{bth} - \sum_k COALDEL_{bkth} \\
 \sum_m \sum_c (1 - wet_c) coal(b, k, t, m) SUM_{ktm} & = \sum_a PROD_{kta} \\
 \sum_c PROD_{kta} & \geq dem_{at} \\
 k \text{ delivering } a &
 \end{array} \right.
 \end{aligned} \tag{23}$$

This gives a *mixed integer program* with *binary variables* *MIXPRES* since a maximal number of different mixes for each plant at each time period must be imposed. This program can also be solved by GAMS/OSL.

At the first iteration, there are 481 rows and 556 columns. Each iteration adds 15 constraints (15 “rows”) and 30 variables (“columns”) to the main problem (See table 8). The new variables correspond to the 15 new mixes binary variables and

the to 15 new mixes utilization levels. The new constraints correspond to the computation of the *SUM* variables corresponding to the 15 new mixes.

#### 4.3. THE GLOBAL PROCESS

The global process includes 3 levels of inner loops in GAMS/OSL:

- the first loop on the *main iterations*,
- the second loop on the *time periods*,
- and the third loop on the *plants*.

It can be summarized by the following procedure:

```
Initialize p(c,k,t) as the sum of the unit production cost
and the delivery price of coal assuming that the coal must be bought;
For m = 1, 2, etc..
{
  For t = 1 to 3
  {
    For k = 1 to 5
    {Determine the optimal mix for plant k, time t
      when using p(c,k,t) as objective coefficients
      by solving the subproblem SP(c,k,t);
      Let coal(c,k,t,m) be the solution obtained;
    }
  }
  Determine the optimal coal purchases, mix use level and demand
  satisfaction planning by solving the main problem MP
  using MIP solveur GAMS/OSL;
  Update p(c,k,t) to the sum of the production unit cost
  and the dual variable of the balance equation for coal c
  at plant k for time period t;
}
```

The *stopping criterion* is as follows: stop if there are no columns pricing out. Before analyzing these results, let us stress the fact that the procedure we propose is *heuristic*. In fact, with an alternative organization of the master problem and subproblem hierarchy, we can achieve a better objective function value (See Section 5).

Let us conclude this section with a few explanations on the “*dual prices*” used in the column generation. In fact, our master problem is a *mixed integer problem* due to the MIXPRES binary variables. It is well know in mathematical programming that the dual information is only valid for a *linear program*. We explain how this dual information is generated. We have solved this MIP problem using GAMS/OSL. After that the solver has found the optimal solution, GAMS fixes the levels of the integer variables and reruns the model as an LP model to obtain the *dual variables values*. As an alternative heuristic, one could also use the *dual prices of the LP relaxation of the whole MIP problem*. We refer to Section 5 for



the comparison of the results of the two possible choices for generating the dual information.

## 5. NUMERICAL RESULTS

Table 8 presents the successive objective function values for the main program, the execution time for GAMS/OSL and the size of the main program.

| Iteration | Objective value (€) | Execution time (sec) | Number of rows | Number of columns |
|-----------|---------------------|----------------------|----------------|-------------------|
| 1         | 91 569 578.20       | 3                    | 481            | 556               |
| 2         | 79 432 349.83       | 4                    | 496            | 586               |
| 3         | 75 029 945.98       | 11                   | 511            | 616               |
| 4         | 71 579 163.48       | 48                   | 526            | 646               |
| 5         | 69 789 373.33       | 49                   | 541            | 676               |
| 6         | 68 615 626.73       | 522                  | 556            | 706               |
| 7         | 68 514 039.00       | 52                   | 571            | 736               |
| 8         | 68 513 503.25       | 409                  | 586            | 766               |
| 9         | 68 428 567.03       | 119                  | 601            | 796               |
| 10        | 68 393 561.65       | 4151                 | 616            | 826               |
| 11        | 68 388 314.23       | 664                  | 631            | 856               |
| 12        | 68 361 242.08       | 4403                 | 646            | 886               |
| 13        | 68 358 137.05       | 494                  | 661            | 916               |
| 14        | 68 349 665.80       | 2412                 | 676            | 946               |
| 15        | 68 347 547.93       | 1865                 | 691            | 976               |
| 16*       | 68 346 480.40       | 3184                 | 706            | 1006              |

\* Stopping criterion satisfied.

TABLE 8. Evolution of the objective function of the MP

Two conclusions can be taken from this table. The first one is the *important objective function decrease* (about 2 500 000 € per iteration) during the *first five iterations*. This global cost reduction continues at a lower rate during the following iterations. Secondly, the heuristic *converges in only 16 mains iterations*. Recall also that since we use a heuristic method, the solution can be suboptimal.

Now, let us say a few words about *two other heuristics* that were suggested to us to solve the problem. The first one is a variant of our column generation method that was suggested to us by professor Yves Pochet from CORE, Université Catholique de Louvain, Belgium. Since the only difficulty in the main program is the fact that there are binary variables ( $COALPRES_{cktm}$ ) in a product with another variable  $SUM_{ktm}$  (See 21), one can, for the main program, *fix the coals*

*present in the mix instead of fixing their proportion in the mix.* This implies, in our method that we send from the lower to the upper level not the *COAL* variables but only the *COALPRES* variables. This also implies that we must impose, in the main program, *all the quality constraints on the mix* listed in appendix A.

Table 9 presents the evolution of the objective function value, the execution time for GAMS/OSL and the size of the main program for this second heuristic method.

| Iteration | Objective value (€) | Execution time (sec) | Number of rows | Number of columns |
|-----------|---------------------|----------------------|----------------|-------------------|
| 1         | 89 378 751.35       | 3                    | 1124           | 796               |
| 2         | 76 709 252.58       | 23                   | 1782           | 1066              |
| 3         | 75 052 581.23       | 18                   | 2440           | 1336              |
| 4         | 70 284 765.43       | 215                  | 3098           | 1606              |
| 5         | 68 484 730.45       | 49                   | 541            | 676               |
| 6         | 68 458 216.33       | 1497                 | 4414           | 2146              |
| 7         | 68 408 431.40       | 543                  | 5072           | 2416              |
| 8         | 68 374 826.85       | 1010                 | 5730           | 2686              |
| 9         | 68 362 017.08       | 599                  | 6388           | 2956              |
| 10        | 68 353 850.75       | 8165                 | 7046           | 3226              |
| 11        | 68 348 399.20       | 10803                | 8362           | 3766              |
| 12        | 68 345 582.43       | 10803                | 646            | 886               |
| 13        | 68 342 647.50       | 10803                | 9020           | 4036              |
| 14*       | 68 341 879.48       | 7575                 | 9678           | 4306              |

\* Stopping criterion satisfied.

TABLE 9. Evolution of the objective function of the MP for the second method

A *slightly better solution* (68 341 879.48 € instead of 68 346 480.40 €) was obtained after 14 main iterations but *it took much longer* (14 hours 28 min instead of 5 hours 6 minutes for our method). This increase in the execution time is due to the fact that the size of the main problem is increased (compare the last two columns of table 8 and 9).

From this example, one conclude that the second method is not competitive since it gives a similar solution in much more time. This example also points out the fact that our procedure is heuristic since the second heuristic gives a *slightly better solution*.

Finally, we present a *third heuristic* that was suggested to us by professor Etienne Loute from Facultés Universitaires Saint Louis, Brussels, Belgium. The only difference with the first method is the choice of the dual information to update the coal delivery prices for the subproblem. Instead of using the dual information

given at the optimal solution of the MIP version of the master problem, we solved the relaxed version of this master problem and used the dual information of this RMIP problem. Table 10 presents the evolution of the objective function value, the execution time for GAMS/OSL for the RMIP and for the MIP version of the master program.

| Iteration | Objective value (€) | Execution time RMIP (sec) | Execution time MIP (sec) |
|-----------|---------------------|---------------------------|--------------------------|
| 1         | 91 569 578.20       | 2                         | 2                        |
| 2         | 79 432 349.83       | 2                         | 4                        |
| 3         | 75 029 945.98       | 2                         | 19                       |
| 4         | 72 105 817.88       | 2                         | 24                       |
| 5         | 68 835 693.58       | 3                         | 1071                     |
| 6         | 68 503 629.70       | 2                         | 468                      |
| 7         | 68 433 685.88       | 3                         | 1 243                    |
| 8         | 68 400 568.13       | 2                         | 75                       |
| 9         | 68 379 477.03       | 2                         | 624                      |
| 10        | 68 358 511.35       | 2                         | 270                      |
| 11        | 68 349 689.18       | 2                         | 390                      |
| 12        | 68 349 689.18       | 2                         | 385                      |
| 13        | 68 348 287.88       | 2                         | 2458                     |
| 14*       | 68 346 804.38       | 2                         | 555                      |

\* Stopping criterion satisfied.

TABLE 10. Evolution of the objective function of the MP for the third method

This third solution method is close to the standard column generation technique. At the end of the procedure, the integrality of variables for the main problem is restored as follows: after solving the LP-relaxation, a branch and bound procedure is performed with these columns.

A *slightly expansive solution* (68 346 804.38 € instead of 68 346 480,40 €) was obtained after 14 main iterations but in *less time* (2 hours 10 minutes instead of 5 hours 6 minutes with our method). All these run times were in fact acceptable to the industry since we determine the planning for three months. The solution procedure was presented to the industrial and helped both. On one side, it helps the industrial to discover the limiting constraints of its production problem. For example, the industrial was not aware that the number of entering gates is an important constraint for the solution. In fact, if we allow as many coal type as we want in the mix, the production costs decreases. On the other side, our heuristic has helped the industrial to decrease its production cost for the three months.

## 6. CONCLUSIONS

In this paper, we have formulated and solved a real word problem. It concerns the integration of the distribution and mixing of coals to satisfy the demand of coke of a set of customers throughout a planning horizon. The modelization of the problem implies a great number of constraints with binary variables. The binary variables come from the limitation of the number of coal in the mixes at each plant at each time period and from the limitation of the number of mixes used for each plant and each time period. Some of the constraints were non linear. To solve this difficult problem, we have presented three heuristics based on the idea of the column generation technique.

The implementation of our method has helped the industrial company to reduce its production cost.

In the future, we intend to add new quality constraints on the mix which are non linear and that can not be linearized by our present heuristic method. It concern upper and lower bound on the Gisseler Fluidity which is a non linear function of the coals quantities in the mix.

## ACKNOWLEDGMENTS

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## APPENDIX A. QUALITY CONSTRAINTS ON THE MIX

- maximal ash rate of the mix:

$$\sum_c ash_c multash COAL_{cktm} \leq \min_{a \text{ delivered by } k} maxash_a SUM_{ktm}$$

- minimal sulfur rate of the mix:

$$\sum_c sulf_c multisulf COAL_{cktm} \geq \max_{a \text{ delivered by } k} minsulf_a SUM_{ktm}$$

- maximal sulfur rate of the mix:

$$\sum_c sulf_c multisulf COAL_{cktm} \leq \min_{a \text{ delivered by } k} maxsulf_a SUM_{ktm}$$

- maximal alkali rate of the mix:

$$\sum_c alk_c multalk COAL_{cktm} \leq \min_{a \text{ delivered by } k} maxalk_a SUM_{ktm}$$

- minimal and maximal volatiles rate of the mix:

$$minvol SUM_{ktm} \leq \sum_c vol_c COAL_{cktm} \leq maxvol SUM_{ktm}$$

- minimal Low Volume rate of the mix:

$$\sum_{c \text{ is a Low Volume}} COAL_{cktm} \geq \max_{a \text{ delivered by } k} minlv_a SUM_{ktm}$$

- maximal Low Volume rate of the mix:

$$\sum_{c \text{ is a Low Volume}} COAL_{cktm} \leq \min_{a \text{ delivered by } k} maxlv_a SUM_{ktm}$$

- minimal and maximal Mid Volume rate of the mix:

$$minmv SUM_{ktm} \leq \sum_{c \text{ is a Mid Volume}} COAL_{cktm} \leq maxmv SUM_{ktm}$$

- maximal rate for Soft coal in the mix:

$$\sum_{c \text{ is a Soft Coal}} COAL_{cktm} \leq maxsoft SUM_{ktm}$$

- maximal rate for Australian coal in the mix:

$$\sum_{c \text{ is an Australian coal}} COAL_{cktm} \leq \max_{aus} SUM_{ktm}$$